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MATHEMATICAL MODELLING OF NONLINEAR FLUID FLOW IN HETEROGENEOUS POROUS MEDIA

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ABSTRACT

The paper carries out an extensive mathematical modeling of nonlinear flow of fluids through heterogeneous porous medium in the framework of Forchheimer-like inertial effects and the space dependence of permeability. Based on the most rudimentary laws of mass conservation and momentum balance, an extended Darcy-Forchheimer framework is developed, which characterizes the incompressible flow in a rigid and isotropic porous material. Governing equations are non-dimensionalized based on the use of relevant characteristic scales and subsequently important dimensionless parameters quantifying viscous resistance, inertial nonlinearity, and medium heterogeneity are identified. By using similarity transformations, the resulting nonlinear partial differential equations are brought down to a first order nonlinear ordinary differential equation. In the case of weakly heterogeneous porous media, the effect of permeability changes on the velocity field is studied by using a perturbation methodology. The resulting boundary-value problem is numerically solved in a shooting technique together with a fourth-order Runge-Kutta method framework. Numerical and graphical findings indicate that augmenting the impact of inertia substantially improves the growth in velocity whereas the heterogeneity of permeability slows down the flow and changes a pressure gradient. The paper emphasizes the very important relationship between nonlinearity and heterogeneity in flow behavior, and offers a strong theoretical basis to the study of complex transport processes in porous media of subsurface environments and industrial materials.

Keywords: Nonlinear porous media flow, Darcy–Forchheimer model, Heterogeneous permeability, Similarity transformation, Non-dimensionalization, Numerical analysis.

Introduction:

Fluid Transport through porous media is a basic transport phenomenon that is extremely important in a large variety of natural and industrial processes, such as groundwater hydrology, petroleum reservoir engineering, chemical reactors, geothermal energy recovery, and environmental cleanup [1]. The forecasting and management of fluid flow in porous frameworks are required in the optimization of resource recovering, efficient-designing of filtration systems, and the evaluation of transport of contaminants beneath the surface [2]. Typically, it is a difficult and significant research problem because of the complicated geometry of porous materials and the presence of multiple scales in the flow processes to create accurate and reliable mathematical models [3].

The law of classical Darcy has been used long as the basis of the modeling of flow in porous media in low-velocity conditions [4]. The linear association between the pressure gradient and the speed of flow is applicable whenever the viscous forces prevail and the inertial forces are insignificant. Experimental and field observations have however shown that, Darcy law fails when the velocity of flow increases and the Reynolds numbers between the pores are no longer small or when the porous medium is structurally irregular. In this case, nonlinear behavioural deviations will occur, and the nonlinear effects of inertia will be required in order to describe the flow dynamics correctly [5].

In most realistic problems, porous media are natural heterogeneities that have spatial variation in permeability, porosity and connectivity between pores that are caused by geological layering, sediment deposition, compaction or manufacturing processes [6]. These heterogeneities add further complexity to the flow behavior so that there are non-uniform velocity fields and non-linear coupling between the pressure gradients and the transport properties. This causes simplistic predictions to be made by the assumption of homogeneous porous media which are not able to include important flow properties of natural and engineered systems [7].

To overcome these drawbacks, longer models like the Darcy Forchheimer model have been suggested to explain the effects of inertia that occurs in moderate and high flow velocities [8]. Added to the momentum equation, a quadratic velocity term makes possible the modeling of other energy losses related to acceleration of the flow and pore-scale turbulence [9]. This nonlinear model together with spatially-dependent permeability and porosity can offer a more realistic model of fluid motion in a heterogeneous porous medium, and is especially useful in studies involving large injection rates, increased recovery of oil, and storage of energy in the underground [10].

Nonlinear flow in non-porous and porous media that are heterogeneous often has a mathematical modeling resulting in highly nonlinear partial differential equations which are hard to solve analytically [11]. Transformation to similarity and non-dimensionalisation methods provide useful means to simplify such equations and detect important dimensionless parameters that control the flow [12]. These techniques are not only useful in making the computation numerically accessible, but offer more physical understanding of the comparative amounts of viscous resistance, inertial influences, or medium heterogeneity [13].

The purpose of the current research is the creation of the generalized mathematical model of nonlinear flow of fluids through heterogeneous porous media taking into consideration Forchheimer-type inertial effects and spatial permeability variability [14]. The effects of nonlinearity and heterogeneity on velocity distribution and pressure gradients are studied through systematic non-dimensionalization, similarity analysis and numerical solution of resultant nonlinear ordinary differential equations [15]. The findings lead to a more theoretical appreciation of the complex flow behavior in porous structures and to the basis of better modeling of practical subsurface and industrial flow systems [16].

Governing Equations:

The mathematical modeling of nonlinear heterogeneous porous media fluid flow is based on the provisions that integrate the profound postulates of mass conservation and momentum balance with the relevant constitutive provisions that ensure that the medium heterogeneity and nonlinear flow influences are factored in the mathematical model [17]. The porous medium is suggested to be rigid, isotropic and totally filled with an incompressible fluid. Based on these assumptions, the conservation of the mass of incompressible flow of a porous medium implies that the net volumetric flux of the fluid into and out of any representative elementary volume should be equal to zero. This situation is stated by the continuity equation [18].

$$\nabla \cdot v = 0 \tag{1}$$

where v is the Darcy velocity, which is a unit of volumetric flow rate of the porous medium, per unit area of the cross-sectional area of the porous medium. Fluid density is constant and because of this, Equation (1) guarantees that there is no buildup or loss of mass in the porous structure and is a basic restriction on the velocity field in the flow space [19].

When fluid flow is at low flow velocities, the flow of fluids in porous media can be explained well by the application of the Darcy law, which presupposes that the fluid velocity is linearly proportional to the pressure gradient. But at higher flow velocity or when inertial effects are caused by the geometry of the pore, this linear relationship is inappropriate. In order to model such nonlinear behavior, DarcyForchheimer model is adopted. The nonlinear flow in a heterogeneous porous medium is represented as a generalized momentum equation which is represented as follows [20].

$$-\nabla p = \frac{\mu}{K(x)} v + \rho\beta \mid v \mid v \tag{2}$$

in which p represents fluid pressure, μ is the dynamic viscosity, rho is fluid density, and K(x) is spatially varying permeability of the porous material. The first term on the right-hand side is a viscous resistance due to the interaction between fluid and solid affixed medium, and the second term is an inertial resistance, with β a Forchimmer coefficient. The nonlinear term $\rho\beta|v|v$ becomes increasingly significant at higher velocities and reflects additional energy losses due to flow acceleration and pore-scale inertial effects [21].

Natural and engineered systems that are subject to porous media can be found with spatial variations in permeability and porosity because of the fact that the pore structure, grain size distribution, and material composition differ. In order to include these effects, the permeability is modelled as a spatially varying function which is provided by

$$K(x) = K_0\big(1 + \varepsilon f(x)\big) \tag{3}$$

In which K_0 is the reference permeability of the medium, ε is a small dimensionless heterogeneity parameter and f(x) is a limited function that represents the spatial distribution of permeability variations [22]. This formulation allows to investigate the effects of heterogeneity in a systematic way and maintain mathematical tractability. Equally, porosity with differences is depicted as

$$\phi(x) = \phi_0\big(1 + \delta g(x)\big) \tag{4}$$

In which ϕ 0 denotes the reference porosity, δ is the parameter of heterogeneity of porosity, and g(x) is a confined function that defines the variation of porosity. Even though porosity differences are not presented in the momentum equation, they are important in transports and energy losses in the porous medium [23].

To make the analysis simpler and concentrate on the key nonlinear and heterogeneity effects, it is assumed that the flow is steady and one dimensional along the xxx-direction. Assuming this, the velocity field becomes, where is the constant magnetic field.

$$v = v(x) \tag{5}$$

When (5) is replaced into the momentum equation (2), the vector equation simplifies into the scalar equation [24].

$$-\frac{dp}{dx} = \frac{\mu}{K(x)}\, v + \rho\beta v^2 \tag{6}$$

which is a tradeoff between the pressure gradient that acts on the flow and the viscous and inertial resistance provided by the heterogeneous porous medium. The inclusion of the nonlinear quadratic velocity term points out the nonclassicality of the Darcy flow and the significance of inertial effects in high-velocity cultures [25].

Nonlinear fluid flow in porous media involves the energy dissipation due to the viscous friction as well as inertial losses. In the case of such flows, the total energy dissipation per unit volume may be written as.

$$D = \frac{\mu v^2}{K(x)}\, v + \rho\beta v^3 \tag{7}$$

The former involves viscous energy losses associated with friction between the fluid and the solid matrix and the latter involves additional loss of dissipation by inertial effects. The relation offers significant understanding of the effectiveness of fluid transport in heterogeneous porous media and is especially applicable in the use of enhanced oil recovery, filtration, and subsurface energy systems.

Non-Dimensionalization and Similarity Analysis:

The non-dimensionalization is used to bring the governing equations to a more general and analytically manageable form, by using relevant characteristic scales. where L r represents a characteristic length scale of the porous medium, V_0 a characteristic velocity scale, and P_0 a characteristic pressure scale. The dimensionless variables are brought in as

$$x^* = \frac{x}{L}, v^* = \frac{v}{V_0}, p^* = \frac{p}{p_0} \tag{8}$$

These change of variables makes the physical variables normal and enables the governing equations to be related to the dimensionless parameters which describe the relative significance of the viscous, inertial, and heterogeneity effects. Replacing the dimensionless variables in Equation (8) in the dimensional momentum equation (6) results in

$$-\frac{dp^*}{dx^*} = Av^* + Bv^{*2} \tag{9}$$

where the dimensionless parameters A and B are defined as

$$A = \frac{\mu L}{K_0 P_0}, B = \frac{\rho \beta V_0 L}{P_0} \tag{10}$$

A is the fractional contribution of a viscous resistance value because of the porous matrix and B is the power of inertial influence value because of nonlinear flow conduct. These parameters are at the centre of identifying the flow regime in the porous medium.

In order to add the permeability heterogeneity to the non-dimensional framework, the spatial variation of permeability is integrated in the viscous resistance parameter. In this respect, Ais was modified in terms of the dimensionless spatial coordinate x^* and is expressed as

$$A(x^*) = \frac{A_0}{1 + \varepsilon f(x^*)} \tag{11}$$

where $A_0 = \mu L/(K_0 P_0)$ represents the viscous resistance for a homogeneous porous medium, and $\varepsilon f(x^*)$ accounts for spatial heterogeneity. This formulation enables direct assessment of how permeability variations influence the velocity field.

In another attempt to simplify the governing equation and perhaps find self-similar solutions, a similarity transformation is introduced. Let

$$\eta = x^*, v^* = F(\eta) \tag{12}$$

and η is the similarity variable and F(η) is the dimensionless velocity function. Replacing Equation (12) in the dimensionless momentum equation (9) results in

$$-\frac{dP}{d\eta} = A(\eta)F + BF^2 \tag{13}$$

and P(η) = in the dimensionless form of pressure as a product of the similarity variable. To differentiate Equation (13) with respect to η we have.

$$\frac{d^2P}{d\eta^2} = -A'(\eta)F - A(\eta)F' - 2BFF' \tag{14}$$

When the terms of pressure gradient are removed in the governing equations, they lead to a first order nonlinear ordinary differential equation, which is the velocity distribution:

$$A(\eta)F' + 2BFF' + A'(\eta)F = 0 \tag{15}$$

The basic similarity equation in nonlinear flow in heterogeneous porous media is called equation (15). It directly models the joint action of viscous resistance, inertial nonlinearity and spatial heterogeneity on the velocity field.

For weakly heterogeneous porous media, where the heterogeneity parameter satisfies $\varepsilon \ll 1$, the function $A(\eta)$ can be approximated using a first-order Taylor expansion as

$$A(\eta) \approx A0\big(1 - \varepsilon f(\eta)\big) \tag{16}$$

Substituting Equation (16) into Equation (15) yields the perturbed nonlinear ordinary differential equation

$$(A_0 + 2BF)F' - \varepsilon A_0 f(\eta)F' - \varepsilon A_0 f'(\eta)F = 0 \tag{17}$$

This equation points to the fact that the heterogeneity of permeability changes the velocity gradient and adds extra coupling between the velocity field and the spatial heterogeneity function.

The similarity equation is obtained by solving with physically significant boundary conditions. At the inlet or reference point $\eta=0$ the velocity is assumed to zero whereas many downstream the velocity tends to converge to the characteristic velocity scale. In this regard, the boundary conditions are are

$$F(0) = 0, F(\infty) = 1 \tag{18}$$

These conditions guarantee the nonlinear flow complete development in the porous media, even in the no-flow or the low-velocity regimes.

Numerical Solution and Graphical Analysis:

Equation (17) with the boundary conditions stated in Equation (18) is a nonlinear, heterogeneity-dependent, ordinary differential equation that cannot be solved analytically since it is a boundary-value problem. As a result, the velocity distribution in the porous medium is acquired using a numerical method in order to get approximate solutions. A shooting method coupled with a fourth-order RungeKutta integration scheme is used to solve the problem. In this type of approach, a problem with a boundary value is turned into a problem with an initial value by refining the unknown initial slope until the far field boundary condition is achieved with a reasonable level of accuracy.

The mathematical calculations are done to investigate how the nonlinearity parameter B and the heterogeneity parameter ε affect the dimensionless velocity function F(eta). These parameters respectively denote inertial effects of nonlinear flow and the level of spatial variation of permeability.

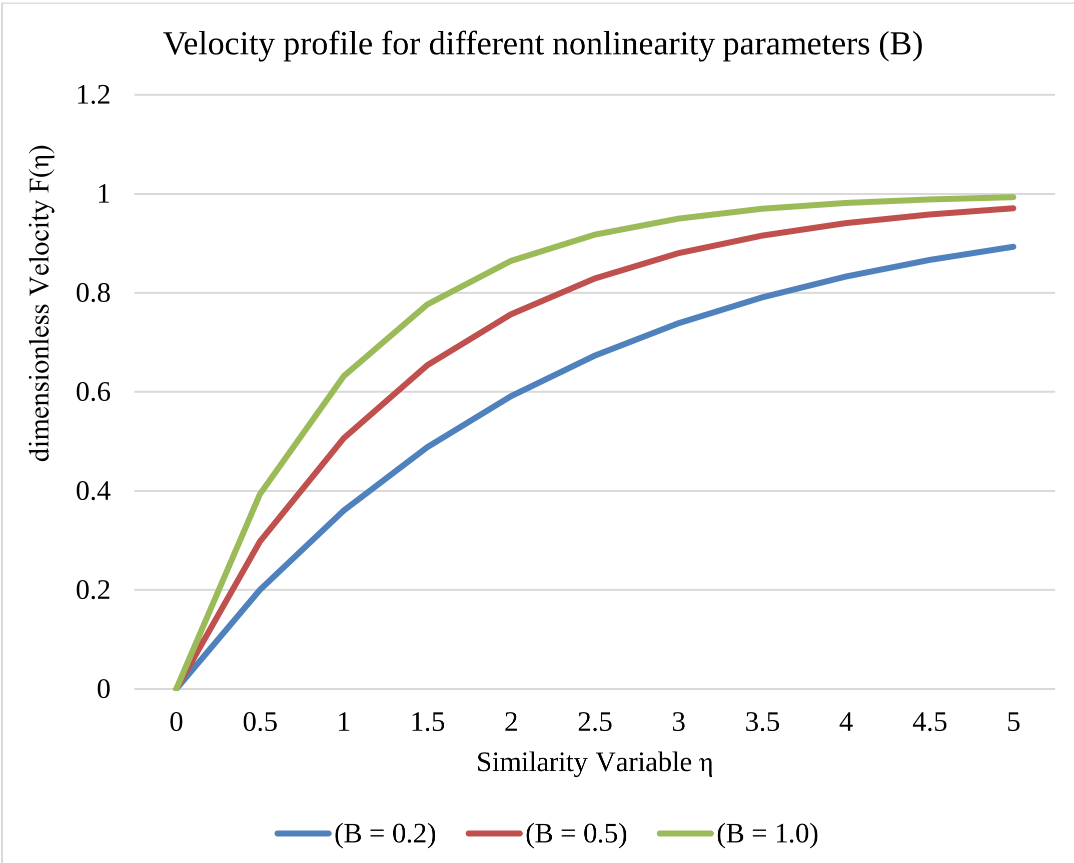

Figure 1: Velocity profile for different nonlinearity parameters B

Figure 1 represents the velocity profiles of various values of nonlinearity parameter B. It is noted that the velocity increases faster with increasing B with respect to similarity variable η and that at the smaller distance the velocity approaches its common-sense value. This trend provides that the stronger the inertial effects, the better the momentum transfer will be and lead to a complete velocity distribution. A smaller value of B will cause the viscous resistance to predominate and the velocity increases more gradually.

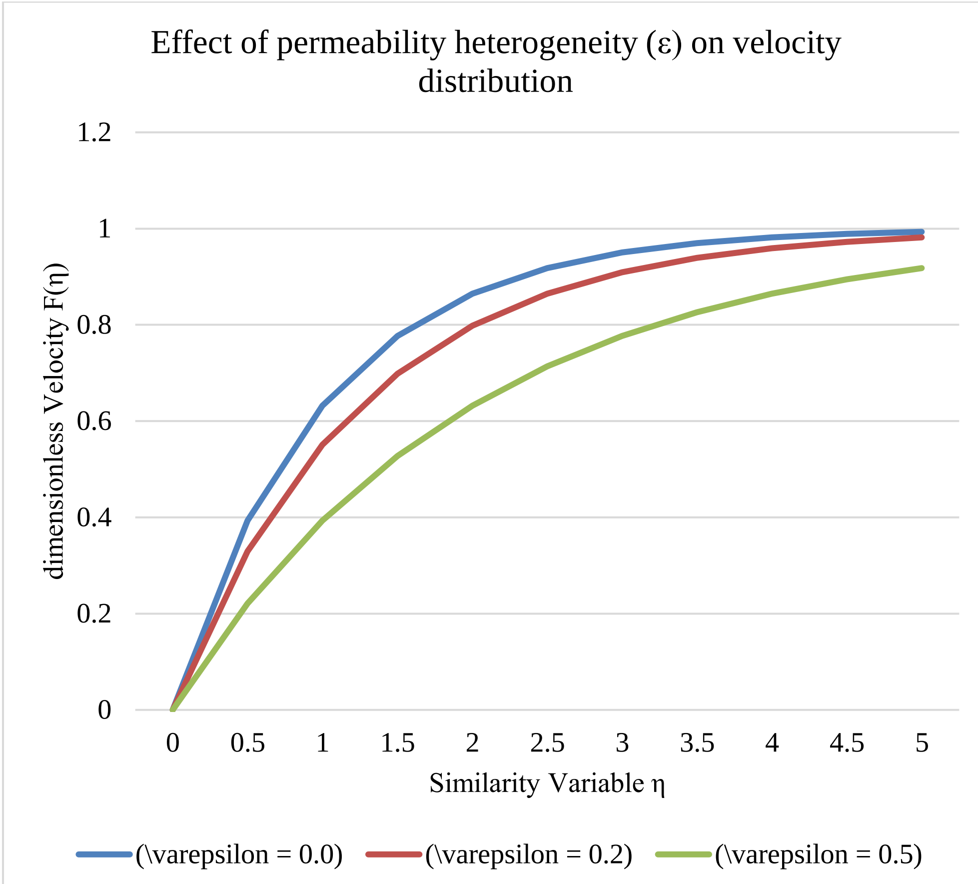

Figure 2: Effect of permeability heterogeneity parameter ε on velocity

The impact of permeability heterogeneity on the velocity distribution is illustrated in Figure 2 with a series of values of the heterogeneity parameter ε. In the case of 0, the porous medium is homogeneous, and the velocity rises gradually to its asymptotic value. It amplifies deviation to a homogeneous case as the value of ε increases. Increased heterogeneity results in decreased velocity levels especially in areas related to low permeability which is a retarding effect of spatial variability on fluid motion.

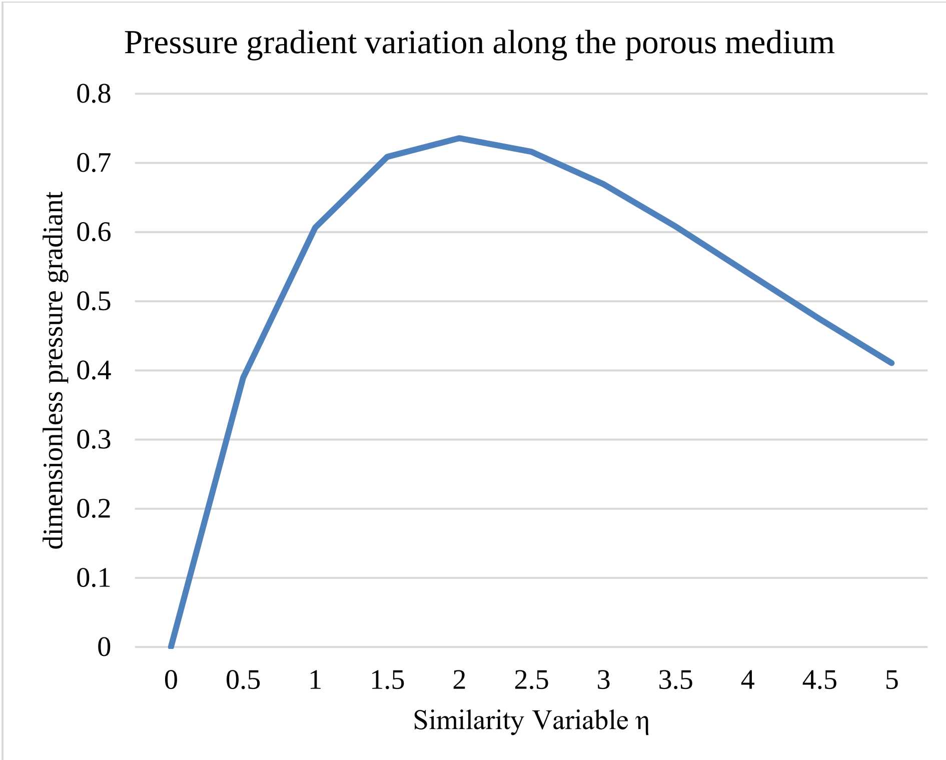

Figure 3: Pressure gradient variation along the porous medium

The dimensionless pressure gradient variation along the porous medium is illustrated in figure 3. The dependence of the pressure gradient on the similarity variable is nonlinear, directly proportional to the similarity variable, but slowly decreasing as the flow enters a fully developed regime. This nonlinear phenomenon verifies the importance of Forchheimer effects, in particular in heterogeneous porous media where inertial losses make a significant contribution to pressure dissipation.

In general, it is evident that the numerical and graphical findings indicate that nonlinearity as well as heterogeneity are important factors in determining the flow characteristics. The interaction of the viscous resistance, inertial effects, and a variation of the permeability plays a major role in the development of velocities and distribution of pressure and one should always consider the use of nonlinear models in examining the flow through realistic porous structures.

Conclusion:

The current research paper has come up with a detailed mathematical framework to examine the non-linear fluid flow in the heterogeneous porous materials by taking into consideration Forchheimer-type inertial forces and the spatial variation in the permeability. Based on the main conservation laws, governing equations were non-dimensionalized and simplified by similarity transformations to nonlinear ordinary differential equations, which made it possible to study the behavior of the flow systematically. Numerical results derived with a shooting scheme of fourth-order Runge-Kutta scheme show that inertial effects play a crucial role in velocity development and permeability heterogeneity works in retarding the flow and changing the pressure gradients. These findings strongly suggest that the pressure gradient is highly nonlinear in nature, and therefore, it is important to consider the Forchheimer effects in the real-world porous systems. Altogether, the research proves that nonlinearity and heterogeneity are very important factors in the flow regulation, and they should be included to guarantee the proper prediction of the transport processes within the framework of the subsurface and industrial use of porous media.

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